Locus Surfaces and Linear Transformations when Fixed Point is at an Infinity

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Abstract

We extend the locus problems discussed in [9], [10] and [12], for a quadric surface when the fixed point is at an infinity. This paper will benefit those students who have backgrounds in Linear Algebra and Multivariable Calculus. As we shall see that the transformation from a quadric surface $\sum$ to its locus surface Δ is a linear transformation. Consequently, how the eigenvectors are related to the position of the fixed point at an infinity will be discussed.

1 Introduction

In *[10]*, we consider the following:

Original problem: *We are given a fixed point A and a generic point C on a surface Σ. We let the line l pass through A and C and intersect a well-defined D on Σ, we want to determine the locus surface generated by the point E, lying on CD, which satisfies $\overrightarrow{ED} = s\overrightarrow{CD}$, where s is a real number parameter.*

We call the point D to be the antipodal point of C, and we often write the locus point as $E = sC+(1-s)D$ in our discussions with no confusions. We provide proofs in this paper, where the discussions originated from *[12]*, how the locus surface for a quadric shall behave when the fixed point $A = (\rho\cos u_0 \sin v_0, \rho\sin u_0 \sin v_0, \rho\cos v_0)$ is at an infinity; we remark that $\rho \to \infty$ and the point A depends on the angles u_0 and v_0. We recall from *[12]* that the locus surfaces, when the surface Σ is an ellipsoid or an hyperboloid with two sheets, we have found the exact expressions for the antipodal point D_{inf} corresponding to point C on Σ when A is at an infinity. In Sections 2 and 3, we discuss how the locus problems for an ellipsoid or a hyperboloid with two sheets can be described as a linear transformation and how their respective eigenvectors and eigenvalues are related to the behaviors of the corresponding locus surfaces. In Section 4, we give a geometric descriptions for the locus surfaces when the parameter s is a large number, including when $s \to \infty$.

2 The Locus Surface When Fixed Point is at an Infinity

If Σ is the quadric surface $F(x, y, z) = 0$ we recall from *[10]* how we find the locus surface of Σ with respect to the fixed point $A = (x_0, y_0, z_0)$. We represent a generic point on Σ as

$$C = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}. \tag{1}$$

We used Vieta's formulas to calculate the coordinates of point D, denoted by (x_1, y_1, z_1), which is the antipodal point of C and is the intersection between the quadric Σ and the line l passing through A and C. The point $E = sC + (1 - s)D$, which is denoted by (x_e, y_e, z_e), generates the locus surface that we will explore in this paper. We remark that once the fixed point A is chosen, since A and C together determine the point E, the locus surface is thus fixed too. We write the locus surface as follows:

$$\Delta_A(C) = \begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix} = \begin{bmatrix} s\hat{x} + (1-s)x_1 \\ s\hat{y} + (1-s)y_1 \\ s\hat{z} + (1-s)z_1 \end{bmatrix}.$$

Unless otherwise specified in this paper, we focus on the parameter $s > 1$ in this paper. In what follows, we shall simplify use Δ for a locus surface with no confusion.

We summarize from [12] how we find the locus of Σ with respect to a fixed point A, which is at an infinity.

1. Let the spherical coordinate for the fixed point A be $(\rho \cos u_0 \sin v_0, \rho \sin u_0 \sin v_0, \rho \cos v_0)$. If we define two auxiliary functions, namely

$$k \doteq k(\hat{x}, \hat{y}) = \frac{\hat{y} - y_0}{\hat{x} - x_0}, \text{ and} \tag{2}$$

$$m \doteq m(\hat{x}, \hat{z}) = \frac{\hat{z} - z_0}{\hat{x} - x_0} \tag{3}$$

2. We follow the usual procedure to find the intersection between the line AC and the quadric surface at $D = (x_1, y_1, z_1)$ respectively by adopting the Vieta's formula.

3. Next we let $\rho \to \infty$ to obtain the corresponding intersection point $D_{\inf} = (x_{1\inf}, y_{1\inf}, z_{1\inf})$.

4. The corresponding locus surface, is defined as $E_{\inf} = (x_{e\inf}, y_{e\inf}, z_{e\inf})$ where

$$x_{e\inf} = s\hat{x} + (1-s)(x_{1\inf})$$
$$y_{e\inf} = s\hat{y} + (1-s)(y_{1\inf})$$
$$z_{e\inf} = s\hat{z} + (1-s)(z_{1\inf}).$$

If $A = (\rho \cos u_0 \sin v_0, \rho \sin u_0 \sin v_0, \rho \cos v_0)$. Let us note that (2) and (3) become,

$$k = \frac{\hat{y} - \rho \sin u_0 \sin v_0}{\hat{x} - \rho \cos u_0 \sin v_0}, \text{ and} \tag{4}$$

$$m = \frac{\hat{z} - \rho \cos v_0}{\hat{x} - \rho \cos u_0 \sin v_0}. \tag{5}$$

5. We fix the angles $u_0 \in (0, 2\pi) - \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ and $v_0 \in (0, \pi)$, and let the point A going to infinity in the direction $(\sin v_0 \cos u_0, \sin v_0 \sin u_0, \cos v_0)$. Taking the limit of (4) and (5) when $\rho \to \infty$ we get,
$$k_0 \doteq k(u_0, v_0) = \frac{\sin v_0 \sin u_0}{\sin v_0 \cos u_0} = \tan u_0, \tag{6}$$
and
$$m_0 \doteq m(u_0, v_0) = \frac{\cos v_0}{\sin v_0 \cos u_0} = \cot v_0 \sec u_0. \tag{7}$$

6. By using the followings and substitute into the implicit equation of the quadric, $F(x, y, z) = 0$,
$$\begin{aligned} y &= \hat{y} + k_0(x - \hat{x}), \text{ and} \\ z &= \hat{z} + m_0(x - \hat{x}), \end{aligned}$$
we follow the Vieta's formula to find the x–coordinate of the the antipodal point D'_{inf}, say x'_{1inf}, by calculating the roots of the polynomial
$$p(x) = a_2x^2 + a_1x + a_0,$$
where a_0, a_1 and a_2 are real numbers.

7. For a given s, the locus surface generated by point $E'_{\text{inf}} = sC + (1 - s)D'_{\text{inf}}$ is defined as
$$\Delta'_{\text{inf}}(s, u_0, v_0) = \begin{bmatrix} x'_{e\,\text{inf}} \\ y'_{e\,\text{inf}} \\ z'_{e\,\text{inf}} \end{bmatrix} = \begin{bmatrix} s\hat{x} + (1 - s)x'_{1\,\text{inf}} \\ s\hat{y} + (1 - s)y'_{1\,\text{inf}} \\ s\hat{z} + (1 - s)z'_{1\,\text{inf}} \end{bmatrix}.$$
It is clear that $D_{\text{inf}} = D'_{\text{inf}}$, and therefore $E_{\text{inf}} = E'_{\text{inf}}$, so the locus surfaces $\Delta_{\text{inf}}(s, u_0, v_0) = \Delta'_{\text{inf}}(s, u_0, v_0)$.

3 Locus Surfaces and Linear Transformations

Theorem 1 *Let Σ be a quadric surface, and let $A_{\text{inf}}(u_0, v_0)$ be the fixed point at an infinity in the direction of $(\cos u_0 \sin v_0, \sin u_0 \sin v_0, \cos v_0)$, $C \in \Sigma$ and D_{inf} be the "antipodal" point of C corresponding to $A_{\text{inf}}(u_0, v_0)$ as described in previous sections. Then there exists an affine transformation $\mathcal{A}_D : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathcal{A}_D(C) = D_{inf}$.*

Proof.

Notice that for a general quadric surface, after applying the Vieta's formula to the polynomial $p(x) = a_2x^2 + a_1x + a_0$ when using (6) and (7), we obtain
$$\begin{aligned} x_{1\,\text{inf}} &= -\frac{a_1}{a_2} - \hat{x} \\ y_{1\,\text{inf}} &= \hat{y} + k_0(x_{1\,\text{inf}} - \hat{x}) \\ &= \hat{y} + k_0\left(-\frac{a_1}{a_2} - 2\hat{x}\right) \\ &= -2k_0\hat{x} + \hat{y} - \frac{a_1}{a_2}k_0, \end{aligned}$$

and

$$\begin{aligned} z_{1\,\mathrm{inf}} &= \hat{z} + m_0(x_{1\,\mathrm{inf}} - \hat{x}) \\ &= \hat{z} + m_0\left(-\frac{a_1}{a_2} - 2\hat{x}\right) \\ &= -2m_0\hat{x} + \hat{z} - \frac{a_1}{a_2}m_0. \end{aligned}$$

We therefore can write $D_{\mathrm{inf}} = MC - \frac{a_1}{a_2}b$, where

$$M = \begin{pmatrix} -1 & 0 & 0 \\ -2k_0 & 1 & 0 \\ -2m_0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ k_0 \\ m_0 \end{pmatrix} \tag{8}$$

Corollary. Given $s > 0$, consider same hypothesis as in Theorem 1 and let $E_{\mathrm{inf}} = sC + (1-s)D_{\mathrm{inf}}$. Then the affine transformation

$$\mathcal{A}_E = sI + (1-s)\mathcal{A}_D$$

is such that $\mathcal{A}_E(C) = E_{inf}$, where I is the identity mapping from R^3 to R^3.

Proposition 2 *In Theorem 1, if Σ is the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, then there exists a matrix $L_D^e = \left[l_{ij}^e\right]_{3\times 3}$ such that $L_D^e C = D_{\mathrm{inf}}$.*

Proof.
It follows from the direct calculations in exploration [S1] show that

$$\begin{aligned} x_{1inf} = &\frac{\left(a^2c^2\sin^2(u_0) - b^2c^2\cos^2(u_0)\right)\sin^2(v_0) + a^2b^2\cos^2(v_0)}{\delta}\ \hat{x} \\ &- \frac{2a^2c^2\cos u_0 \sin u_0 \sin^2(v_0)}{\delta}\ \hat{y} \\ &- \frac{2a^2b^2\cos u_0 \cos v_0 \sin v_0}{\delta}\ \hat{z} \\ y_{1inf} = &-\frac{2b^2c^2\cos u_0 \sin u_0 \sin^2(v_0)}{\delta}\ \hat{x} \\ &+ \frac{\left(b^2c^2\cos^2(u_0) - a^2c^2\sin^2(u_0)\right)\sin^2(v_0) + a^2b^2\cos^2(v_0)}{\delta}\ \hat{y} \\ &- \frac{2a^2b^2\sin u_0 \cos v_0 \sin v_0}{\delta}\ \hat{z} \\ z_{1inf} = &-\frac{2b^2c^2\cos u_0 \cos v_0 \sin v_0}{\delta}\ \hat{x} \\ &- \frac{2a^2c^2\sin u_0 \cos v_0 \sin v_0}{\delta}\ \hat{y} \\ &+ \frac{\left(a^2c^2\sin^2(u_0) + b^2c^2\cos^2(u_0)\right)\sin^2(v0) - a^2b^2\cos^2(v0)}{\delta}\ \hat{z} \end{aligned}$$

where $\delta = (a^2c^2\sin^2(u_0) + b^2c^2\cos^2(u_0))\sin^2(v_0) + a^2b^2\cos^2(v_0)$. Matrix L^e_D can be written as

$$\frac{1}{\delta}\left(M - \begin{bmatrix} 0 & 2a^2c^2\cos u_0 \sin u_0 \sin^2(v_0) & 2a^2b^2\cos u_0 \cos v_0 \sin v_0 \\ 2b^2c^2\cos u_0 \sin u_0 \sin^2(v_0) & 0 & 2a^2b^2\sin u_0 \cos v_0 \sin v_0 \\ 2b^2c^2\cos u_0 \cos v_0 \sin v_0 & 2a^2c^2\sin u_0 \cos v_0 \sin v_0 & 0 \end{bmatrix}\right),$$

where M is the 3×3 diagonal matrix of the following entries:

$$\begin{bmatrix} \left(a^2c^2\sin^2(u_0) - b^2c^2\cos^2(u_0)\right)\sin^2(v_0) + a^2b^2\cos^2(v_0) \\ \left(b^2c^2\cos^2(u_0) - a^2c^2\sin^2(u_0)\right)\sin^2(v_0) + a^2b^2\cos^2(v_0) \\ \left(a^2c^2\sin^2(u_0) + b^2c^2\cos^2(u_0)\right)\sin^2(v_0) - a^2b^2\cos^2(v_0) \end{bmatrix}.$$

Corollary. Given $s > 0$, consider same hypothesis as in Proposition 2 and let $E_{\text{inf}} = sC + (1-s)D_{\text{inf}}$. Then the matrix

$$L^e_E = sI + (1-s)L^e_D \tag{9}$$

is such that $L^e_E\, C = E_{\text{inf}}$, and therefore, the locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ is the image of Σ under the linear transformation given by the matrix $L^e_E = [l^e_{ij}]_{3\times 3}$. We may, therefore, call the linear transformation that is associated with the matrix L^e_E to be an antipodal linear transformation.

Proposition 3 *For $s \in \mathbb{R} \setminus \{1\}$, the ellipsoid Σ and locus ellipsoid $\Delta_{\text{inf}}(s, u_0, v_0)$ intersect themselves tangentially at an elliptical curve.*

Proof. The proposition was already proved when point $A = (x_0, y_0 z_0)$ is at infinity on $x - axis, y - axis$ or $z - axis$ respectively in [12], so we can suppose that $u_0 \in (0, 2\pi) - \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ and $v_0 \in (0, \pi)$.

Let us determine the points $C \in \Sigma$ such that

$$C = E_{\text{inf}} = sC + (1-s)D_{\text{inf}},$$

that is, such that $(1-s)C = (1-s)D_{\text{inf}}$. For $s \neq 1$, this implies that $L^e_D\, C = D_{\text{inf}}$, which is consistent with direct calculations in [S2] that L^e_D has the eigenvalue $\mu_1 = -1$ of multiplicity 1 with associated eigenvector $v_1 = [\cos u_0 \tan v_0, \sin u_0 \tan v_0, 1]^t$. In addition, L^e_D has the eigenvalue $\mu_2 = 1$ of multiplicity 2 with the following associated eigenvectors

$$\begin{aligned} v_2 &= \left[\frac{-a^2}{c^2}\sec u_0 \cot v_0, 0, 1\right]^t, \text{ and} \\ v_3 &= \left[-\frac{a^2}{b^2}\tan u_0, 1, 0\right]^t. \end{aligned}$$

The intersection of the plane generated by v_2 and v_3 with the ellipsoid Σ is the elliptical curve $\gamma \doteq \gamma_{a,b,c,u_0,v_0}(x(t), y(t), z(t))$, with

$$\begin{aligned} x &= t, \\ y &= \mp\frac{b^2\cos v_0\ \alpha(t) - \beta(t)}{a^2c^2\sin^2 u_0 \sin^2 v_0 + a^2b^2\cos^2 v_0}, \\ z &= \pm\frac{c^2\sin u_0 \sin v_0\,(b^2\cos v_0\ \alpha(t) + \beta(t))}{b^2\cos v_0\left(a^2c^2\sin^2 u_0 \sin^2 v_0 + a^2b^2\cos^2 v_0\right)} - \frac{c^2}{a^2}\cos u_0\,(\tan v_0)\,t, \end{aligned} \tag{10}$$

where

$$\alpha(t) \doteq \sqrt{\left(\left(-a^2c^2\sin^2 u_0 - b^2c^2\cos^2 u_0\right)\sin^2 v_0 - a^2b^2\cos^2 v_0\right)t^2 + a^4c^2\sin^2 u_0\sin^2 v_0 + a^4b^2\cos^2 v_0}$$

and

$$\beta(t) \doteq b^2c^2\cos u_0 \sin u_0 \left(\sin^2 v_0\right)\, t.$$

Finally, we can verify that the gradient of Σ and $\Delta_{\text{inf}}(s, u_0, v_0)$ are colinear when evaluated at any point on γ.

Proposition 4 *In Theorem 1, if Σ is the hyperboloid with two sheets $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$, then there exists a matrix $L_D^h = \left[l_{ij}^h\right]_{3\times 3}$ such that $L_D^h C = D_{\text{inf}}$.*

Proof. We leave the proof to readers to explore.

Corollary. Given $s > 0$, consider same hypothesis as in Proposition 4, and let $E_{\text{inf}} = sC + (1-s)D_{\text{inf}}$. Then the matrix

$$L_E^h = sI + (1-s)L_D^h$$

is such that $L_E^h C = E_{\text{inf}}$. In other words, the locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ is the image of Σ under the linear transformation given by the matrix $L_E^h = [l_{ij}^h]_{3\times 3}$.

We remark that the exploration [S3] contains an animation to exemplify the result Proposition 3. Analogous to the Proposition 3, we have the following:

Proposition 5 *For $s \in \mathbb{R}^+\backslash\{1\}$, if the hyperboloid Σ and corresponding locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ intersect themselves, they do it tangentially at an hyperbolical curve.*

Proof. The proposition was already proved when point $A = (x_0, y_0 z_0)$ is at infinity on $x - axis, y - axis$ or $z - axis$ respectively in [12], so we may assume that $u_0 \in (0, 2\pi) - \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ and $v_0 \in (0, \pi)$. Let us determine the points $C \in \Sigma$ such that

$$C = E_{\text{inf}} = sC + (1-s)D_{\text{inf}};$$

that is, such that $(1-s)C = (1-s)D_{\text{inf}}$. For $s \neq 1$, this implies that $L_D^h C = D_{\text{inf}}$. Direct calculations in Explorations [S4] and [S5] show that L_D^h has the eigenvalue $\mu_1 = -1$ of multiplicity 1 with associated eigenvector and the eigenvalue $\mu_2 = 1$ of multiplicity 2 with associated eigenvectors

$$v_2 = \left(\frac{a^2}{c^2}\sec u_0 \cot v_0, 0, 1\right) \quad \text{and} \quad v_3 = \left(-\frac{a^2}{b^2}\tan u_0, 1, 0\right).$$

The intersection of the plane generated by v_2 and v_3 with the hyperboloid Σ of two sheets is the hyperbolical curve $\gamma \doteq \gamma_{a,b,c,u_0,v_0}(x(t), y(t), z(t))$, with

$$\begin{aligned}
x &= t,\\
y &= \pm\frac{b^2\cos v_0\ \alpha(t) \mp \beta(t)}{a^2c^2\sin^2 u_0\sin^2 v_0 - a^2b^2\cos^2 v_0},\\
z &= \frac{c^2}{a^2}\cos u_0 \tan v_0\, t \pm \frac{c^2\sin u_0 \sin v_0\,(b^2\cos v_0\ \alpha(t) \mp \beta(t))}{b^2\cos v_0\left(a^2c^2\sin^2 u_0\sin^2 v_0 - a^2b^2\cos^2 v_0\right)},
\end{aligned}$$

where

$$\alpha(t) \doteq \sqrt{\left(\left(a^2c^2\sin^2 u_0 + b^2c^2\cos^2 u_0\right)\sin^2 v_0 - a^2b^2\cos^2 v_0\right)t^2 + a^4c^2\sin^2 u_0\sin^2 v_0 - a^4b^2\cos^2 v_0},$$

and

$$\beta(t) \doteq b^2c^2\cos u_0\sin u_0\left(\sin^2 v_0\right)\, t.$$

The following observation is trivial.

Theorem 6 *If $s \in \mathbb{R}^+\backslash\{1/2\}$, the locus surface $\Delta_{\inf}(s, u_0, v_0)$ for an ellipsoid Σ is also an ellipsoid.*

Proof. We consider the ellipsoid Σ, then it is well known result that Σ is an image under a non-singular linear transformation T from a sphere S. In other words, $\Sigma = T(S)$. Now for $s \in \mathbb{R}^+\backslash\{1/2\}$, the transformation L_E^e is non-singular, and the locus $\Delta_{\inf}(s, u_0, v_0) = L_E^e\left(T(S)\right)$, we see that the locus is the image of the sphere S under a non-singular linear transformation and hence it is an ellipsoid too.

We shall prove that the locus surface for a hyperboloid with two sheets is indeed another hyperboloid with two sheets. We prove the following with the help of [4] due to its complex computations, complete computations can be found in [S4].

Theorem 7 *If $s \in \mathbb{R}^+\backslash\{1/2\}$, the locus surface $\Delta_{\inf}(s, u_0, v_0)$ is also a hyperboloid of two sheets.*

Proof. The implicit equation of the locus surface $\Delta_{\inf}(s, u_0, v_0)$ is given by the quadratic form

$$[x\ y\ z\ 1]\ Q_\Delta^h \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0\,, \tag{11}$$

where

$$Q_\Delta^h = \begin{bmatrix} l_{11}^h & l_{21}^h & l_{31}^h & 0 \\ l_{12}^h & l_{22}^h & l_{32}^h & 0 \\ l_{13}^h & l_{23}^h & l_{33}^h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b^2c^2 & 0 & 0 & 0 \\ 0 & a^2c^2 & 0 & 0 \\ 0 & 0 & -a^2b^2 & 0 \\ 0 & 0 & 0 & a^2b^2c^2 \end{bmatrix} \begin{bmatrix} l_{11}^h & l_{12}^h & l_{13}^h & 0 \\ l_{21}^h & l_{22}^h & l_{23}^h & 0 \\ l_{31}^h & l_{32}^h & l_{33}^h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}.$$

We see from the exploration [S5] that the quadratic form becomes

$$\frac{Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 + J}{\delta} = 0$$

where

$$
\begin{aligned}
A &= \left(\left(4a^2b^2c^4s^2 - 4a^2b^2c^4s + \left(a^2b^2 - b^4\right)c^4\right)\cos^2 u_0 - 4a^2b^2c^4s^2 + 4a^2b^2c^4s - a^2b^2c^4\right)\sin^2 v_0 \\
&\quad + \left(4a^2b^4c^2s^2 - 4a^2b^4c^2s + a^2b^4c^2\right)\cos^2 v_0, \\
B &= \left(8a^2b^2c^4s^2 - 8a^2b^2c^4s\right)\cos u_0 \sin u_0 \sin^2 v_0, \\
C &= \left(8a^2b^4c^2s - 8a^2b^4c^2s^2\right)\cos u_0 \cos v_0 \sin v_0, \\
D &= \left(\left(-4a^2b^2c^4s^2 + 4a^2b^2c^4s + \left(a^4 - a^2b^2\right)c^4\right)\cos^2 u_0 - a^4c^4\right)\sin^2 v_0 \\
&\quad + \left(4a^4b^2c^2s^2 - 4a^4b^2c^2s + a^4b^2c^2\right)\cos^2 v_0, \\
E &= \left(8a^4b^2c^2s - 8a^4b^2c^2s^2\right)\sin u_0 \cos v_0 \sin v_0, \\
F &= \left(\left(\left(4a^2b^4 - 4a^4b^2\right)c^2s^2 + \left(4a^4b^2 - 4a^2b^4\right)c^2s + \left(a^2b^4 - a^4b^2\right)c^2\right)\cos^2 u_0 + \right. \\
&\quad \left. 4a^4b^2c^2s^2 - 4a^4b^2c^2s + a^4b^2c^2\right)\sin^2 v_0 - a^4b^4\cos^2 v_0, \\
J &= \left(\left(\left(4a^4b^2 - 4a^2b^4\right)c^4s^2 + \left(4a^2b^4 - 4a^4b^2\right)c^4s + \left(a^4b^2 - a^2b^4\right)c^4\right)\cos^2 u_0 - \right. \\
&\quad \left. 4a^4b^2c^4s^2 + 4a^4b^2c^4s - a^4b^2c^4\right)\sin^2 v_0 + \left(4a^4b^4c^2s^2 - 4a^4b^4c^2s + a^4b^4c^2\right)\cos^2 v_0, \\
\delta &= \left(\left(\left(4b^2 - 4a^2\right)c^2\cos^2 u_0 + 4a^2c^2\right)\sin^2 v_0 - 4a^2b^2\cos^2 v_0\right)\left(s - 1/2\right)^2.
\end{aligned}
$$

It follows from the hypothesis that $\delta \neq 0$, so the implicit equation of the locus surface can be written as,

$$
[x\ y\ z\ 1]\begin{bmatrix} A & B/2 & C/2 & 0 \\ B/2 & D & E/2 & 0 \\ C/2 & E/2 & F & 0 \\ 0 & 0 & 0 & J \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0\,. \tag{12}
$$

Following [8], we see that

$$
\rho_3 \doteq rank\begin{bmatrix} A & B/2 & C/2 \\ B/2 & D & E/2 \\ C/2 & E/2 & F \end{bmatrix} = 3, \rho_4 \doteq rank\begin{bmatrix} A & B/2 & C/2 & 0 \\ B/2 & D & E/2 & 0 \\ C/2 & E/2 & F & 0 \\ 0 & 0 & 0 & J \end{bmatrix} = 4, \tag{13}
$$

and furthermore we use [3] to compute the determinant of $\begin{bmatrix} A & B/2 & C/2 & 0 \\ B/2 & D & E/2 & 0 \\ C/2 & E/2 & F & 0 \\ 0 & 0 & 0 & J \end{bmatrix}$ to be $-\dfrac{a^6b^6c^6}{2s-1}$, which is negative if $s > \frac{1}{2}$. Hence, we conclude that locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ is another copy of hyperboloid with two sheets. We use the following example to illustrate the relationship between the hyperboloid of two sheets and its corresponding locus.

Example 8 *We consider the hyperboloid with two sheets of* $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0$ *with* $a = 5, b = 4, c = 3, u_0 = \frac{\pi}{6}$ *and* $v_0 = \frac{\pi}{3}$. *We shall find the corresponding locus surface when* $s = 2$ *and the intersecting curve between these two surfaces.*

1. We follow Theorem 6 to find the locus surface when $s = 2$ below:
$$\frac{172\,800}{371}\sqrt{3}Y\left(X - \frac{50}{27}Z\right) + \frac{112464X^2}{371} - \frac{614400XZ}{371} + \frac{51525Y^2}{371} + \frac{6455600Z^2}{3339} - 3600 = 0. \quad (14)$$

2. Next we apply Proposition 5 to find the plane spanned by the eigenvectors $\{v_2, v_3\}$, and then find the intersecting curve between the original surface and its locus surface when $s = 2$. We refer readers to [S7] and [S7.1] for further explorations.

3. It is also worth noting that we can find the intersecting curves between the original hyperboloid with two sheets and its corresponding locus directly without using the concepts of eigenvectors. We follow the idea in [12] to find the intersection between the surface $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0$ and the tangent plane, $T(x, y, z) = \nabla F(x, y, z) \cdot (x - x_0, y - y_0, z - z_0) = 0$, that is passing through the fixed point
$$A = (x_0, y_0, z_0) = (\rho \sin v_0 \cos u_0, \rho \sin v_0 \sin u_0, \cos v_0 \rho \cos v_0),$$
and next we let $\rho \to \infty$. We see from [S5.1] that intersecting curve consist of *four branches*, which are shown in blue curves in Figure 1, and we refer readers to [S5.1] and [S6] for further explorations. We use [3] to plot the locus for the hyperboloid with two sheets (in yellow), the original hyperboloid with two sheets (in red) and the intersecting curve in blue in Figure 1 below.

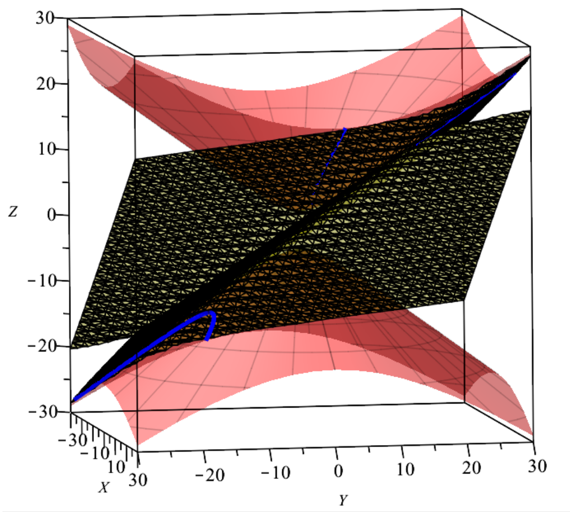

Figure 1. Hyperboloid with two sheets and its locus

3.1 Some observations from the linear transformations

Theorem 9 *Suppose Σ is an ellipsoid of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in the standard form. Let the fixed point A be at an infinity with the fixed (u_0, v_0) direction, and let L_E^e be the linear transformation* *that maps points on Σ to the locus surface with respect to A. For a given s, we have the following observations. We omit the proofs except (2) since they are either direct computations from a CAS or simple observations.*

1. The eigenvalues for L_E^e are $\{2s-1, 1, 1\}$ and the corresponding eigenvectors are as follows:

$$\begin{aligned} v_1 &= [\cos u_0 \tan v_0, \sin u_0 \tan v_0, 1]^t, \qquad (15)\\ v_2 &= \left[\frac{-a^2}{c^2} \sec u_0 \cot v_0, 0, 1\right]^t, \\ v_3 &= \left[-\frac{a^2}{b^2} \tan u_0, 1, 0\right]^t. \end{aligned}$$

We remark that the eigenvectors v_1, v_2 and v_3 are invariant under the parameter s, and invite readers to explore geometrically why $L_E^e(v_2) = v_2$ and $L_E^e(v_3) = v_3$ respectively.

2. When $s = \frac{1}{2}$, the locus surface becomes the elliptical disk bounded by intersecting elliptical curve.

Proof.

Given $C \in \Sigma$, there exist $\alpha_1, \alpha_2, \alpha_3$ such that $C = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$. Then we see

$$E_{\text{inf}} = L_E^e C = \alpha_1 L_E^e v_1 + \alpha_2 L_E^e v_2 + \alpha_3 L_E^e v_3 = \alpha_1(2s-1)v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

For $s = \frac{1}{2}$,

$$E_{\text{inf}} = \alpha_2 v_2 + \alpha_3 v_3$$

that is, E_{inf} is the projection of C in the plane spanned by v_2 and v_3.

3. The intersecting curve between Σ and its locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ lie on the plane P spanned by the eigenvectors v_2 and v_3 corresponding to repeated eigenvalue 1.

4. When the eigenvectors $\{v_1, v_2, v_3\}$ are orthogonal, the locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ can be expressed in its standard form using $\{v_1, v_2, v_3\}$ as the directions of their respective axes.

We state a result, which will be needed later, from [12] when A is at an infinity as follows:

Theorem 10 *For $s > 0$ given, let Σ be the sphere $x^2 + y^2 + z^2 = r^2$, A_1 be at the infinity on the z axis, and $A = (\rho \sin v_0 \cos u_0, \rho \sin v_0 \sin u_0, \rho \cos v_0)$ when $\rho \to \infty$. We denote Δ_1 to be the locus surface of Σ with respect to A_1 and Δ to be the locus surface of Σ with respect to A. If $R_y(v_0)$ represents the rotation by v_0 radians around y–axis, and $R_z(u_0)$ represents the rotation by u_0 radians around z–axis, then $R_z(u_0) \circ R_y(v_0)(\Delta_1) = \Delta$.*

We shall proceed to prove following observation:

Theorem 11 *For arbitrary A (at infinity or not), we denote the solid region with the boundary of $\Delta(\Sigma, A, s)$ by $\underline{\Delta}(\Sigma, A, s)$. If this region is convex, then we have $\Sigma \subset \underline{\Delta}(\Sigma, A, s)$ when $s > 1$.*

Proof. Let C an arbitrary point in Σ with "antipodal" point D. Consider the locus points $E = sC + (1-s)D$ and $E' = sD + (1-s)C$, see Figure 2 below:

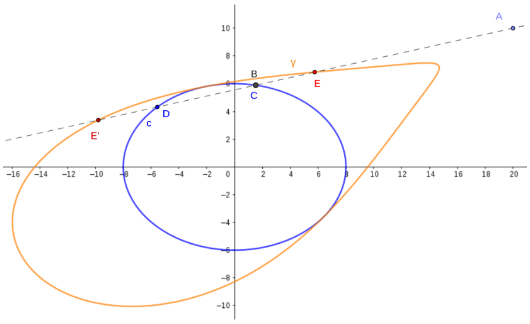

Figure 2. Idea of Theorem 10

A direct calculation shows that

$$C = \frac{s}{2s-1}E + \left(1 - \frac{s}{2s-1}\right)E'.$$

Since $\underline{\Delta}(\Sigma, A, s)$ is a convex set, and $0 < s/(2s-1) < 1$ for $s > 1$, it follows that $C \in \underline{\Delta}(\Sigma, A, s)$.

Corollary *We consider the fixed point A to be at an infinity with the fixed (u_0, v_0) direction, and $s > 1$. Let us denote by*

$$\underline{\Delta}_{\inf}(s, u_0, v_0) = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ F(x, y, z; s, u_0, v_0) \leq 1\right\} \tag{16}$$

the solid ellipsoid with its boundary of $\Delta_{\inf}(s, u_0, v_0)$. Then we have $\Sigma \subsetneq \underline{\Delta}_{x,\inf}$ when $s > 1$.

Theorem 12 *For arbitrary A (at infinity or not) and let us denote by*

$$\underline{\Sigma} = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\}$$

the solid ellipsoid whose boundary is Σ. If $s \in [0, 1]$, Then $\Delta(\Sigma, A, s) \subset \underline{\Sigma}$.

Proof. Let E be an arbitrary point in $\Delta(\Sigma, A, s)$. By construction, there exist points C and D in Σ such that $E = sC + (1-s)D$. Since $\underline{\Sigma}$ is a convex set, it follows that $E \in \underline{\Sigma}$.

Theorem 13 *Suppose Σ is a hyperboloid with two sheets of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ in the standard form. Let the fixed point A be at infinity with fixed (u_0, v_0) and let L_E^h be the linear transformation that maps points on Σ to the locus surface with respect to A. For a given s, we have the followings:*

1. The eigenvalues for L_E^h are $\{2s-1, 1, 1\}$ and the corresponding eigenvectors are as follows:

$$v_1 = [\cos u_0 \tan v_0, \sin u_0 \tan v_0, 1]^t, \tag{17}$$

$$v_2 = \left[\frac{a^2}{c^2} \sec u_0 \cot v_0, 0, 1\right]^t, \tag{18}$$

$$v_3 = \left[-\frac{a^2}{b^2} \tan u_0, 1, 0\right]^t. \tag{19}$$

2. For $s = \frac{1}{2}$.

   (a) If Σ and $\Delta_{\text{inf}}(s, u_0, v_0)$ intersect, the locus surface becomes the hyperbolic plane region bounded by intersecting hyperbolic curve.

   (b) If Σ and $\Delta_{\text{inf}}(s, u_0, v_0)$ do not intersect, the locus surface becomes the plane P spanned by the eigenvectors v_2 and v_3 corresponding to the repeated eigenvalue 1.

3. When Σ and $\Delta_{\text{inf}}(s, u_0, v_0)$ intersect, the intersecting curve between them lie on the plane P spanned by the eigenvectors v_2 and v_3 corresponding to eigenvalue 1.

4. When the eigenvectors $\{v_1, v_2, v_3\}$ are orthogonal, the locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ can be expressed in its standard form using $\{v_1, v_2, v_3\}$ as directions of their respective axes .

3.2 Locus Surface for an ellipsoid when $s \neq \frac{1}{2}$

To complement the result we have shown in Theorem 6, we show here directly that the locus surface for an ellipsoid, when the parameter $s \neq \frac{1}{2}$, under the antipodal linear transformation, is another ellipsoid. We first note that the implicit equation of the ellipsoid $\Delta_{\text{inf}}(s, u_0, v_0)$ is given by the quadratic form

$$(X^*)\, Q^e_\Delta\, (X^*)^T = 0\,, \tag{20}$$

where $X^* = [X\ Y\ Z\ 1]$ and

$$Q^e_\Delta = \begin{bmatrix} l^e_{11} & l^e_{21} & l^e_{31} & 0 \\ l^e_{12} & l^e_{22} & l^e_{32} & 0 \\ l^e_{13} & l^e_{23} & l^e_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b^2c^2 & 0 & 0 & 0 \\ 0 & a^2c^2 & 0 & 0 \\ 0 & 0 & a^2b^2 & 0 \\ 0 & 0 & 0 & -a^2b^2c^2 \end{bmatrix} \begin{bmatrix} l^e_{11} & l^e_{12} & l^e_{13} & 0 \\ l^e_{21} & l^e_{22} & l^e_{23} & 0 \\ l^e_{31} & l^e_{32} & l^e_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}.$$

1. The key is to compute $(X^*)\, Q^e_\Delta\, (X^*)^T = 0$ and collect the coefficients of $x^i y^j$, where i, j are non-negative integers and $i + j = 0, 1, 2$.

2. First, we shall write the 4 by 4 symmetric matrix $Q^e_\Delta = [q_{ij}]$, where $i, j = 1, 2, 3, 4$, explicitly. To begin, we let $\delta^e = 4\left(s - \frac{1}{2}\right)^2 \begin{pmatrix} c^2\left(\cos^2(v_0) - 1\right)\left(a^2 - b^2\right)(\cos u_0)^2 \\ +a^2\left(\left(b^2 - c^2\right)(\cos v_0)^2 + c^2\right) \end{pmatrix}$. Now

we consider

$$\begin{aligned}
q_{11} &= \frac{\left[\begin{pmatrix} 4\left(\cos^2 v_0 - 1\right) c^4 \left(\left(s-\frac{1}{2}\right)^2 a^2b^2 - \frac{b^4}{4}\right)\cos^2 u_0 \\ +4a^2\cos^2 v_0 \left(\left(s-\frac{1}{2}\right)^2 \left(b^4c^2 - b^2c^4\right)\right) \end{pmatrix} \\ +b^2c^4\left(s-\frac{1}{2}\right)^2\right]}{\delta^e}, \\
q_{22} &= \frac{\left[\begin{pmatrix} 4\left(\cos^2 v_0 - 1\right) c^4 \left(\frac{a^4}{4} - \left(s-\frac{1}{2}\right)^2 a^2b^2\right)\cos^2 u_0 \\ +4a^2\cos^2 v_0 \left(a^2b^2c^2\left(s-\frac{1}{2}\right)^2 - \frac{a^2c^4}{4}\right) \\ +\frac{a^2c^4}{4} \end{pmatrix}\right]}{\delta^e}, \\
q_{33} &= \frac{\left[\begin{pmatrix} 4a^2b^2c^2\cos^2 u_0\left(a^2-b^2\right)\left(\cos^2 v_0 - 1\right)\left(s-\frac{1}{2}\right)^2 \\ +4a^2\left(\left(-a^2b^2c^2\left(s-\frac{1}{2}\right)^2 + \frac{a^2b^2}{4}\right)\cos^2 v_0\right) \\ +a^2b^2c^2\left(s-\frac{1}{2}\right)^2 \end{pmatrix}\right]}{\delta^e}, \\
q_{12} &= \frac{a^2b^2c^4 s\left(s-1\right)\left(\cos^2 v_0 - 1\right)\sin u_0\cos u_0}{\delta^e}, \\
q_{13} &= \frac{a^2b^4c^2 s\left(s-1\right)\left(\cos v_0 \sin v_0 \cos u_0\right)}{\delta^e}, \\
q_{23} &= \frac{a^4b^2c^2 s\left(s-1\right)\left(\cos v_0 \sin v_0 \sin u_0\right)}{\delta^e}, \\
q_{14} &= q_{24} = q_{34} = 0, \text{ and } q_{44} = -a^2b^2c^2.
\end{aligned} \tag{21}$$

3. With the help of an CAS, we now compute $\left(X^*\right) Q^e_\Delta \left(X^*\right)^T = 0$ when $s \neq \frac{1}{2}$. If we let

$$\beta = \frac{1}{\begin{array}{c} 4\,c^2\left(\cos\left(v_0\right)-1\right)\left(\cos\left(v_0\right)+1\right)\left(a-b\right)\left(a+b\right)\left(\cos\left(u_0\right)\right)^2 \\ +4\,\left(\left(b^2-c^2\right)\left(\cos\left(v_0\right)\right)^2 + c^2\right)a^2 \end{array}}, \tag{22}$$

then $XQ^e_\Delta X^T = 0$ becomes

$$\beta\left(s-\frac{1}{2}\right)^{-2}\cdot L = 0, \tag{23}$$

where

$$L = \left(\begin{array}{c} 4\,(\cos(v_0) - 1) \left(\left(\begin{array}{c} \left(-\,(s-1/2)^2\, b^2 + 1/4\, Y^2\right) a^4 \\ +\,(s-1/2)^2\, b^2\,(X^2 - Y^2 + b^2)\, a^2 - 1/4\, X^2 b^4 \end{array} \right) c^2 \atop +a^2\,(s-1/2)^2\,(a+b)\, b^2 Z^2\,(a-b) \right) \\ c^2\,(\cos(v_0)+1)\,(\cos(u_0))^2 \\ +8\, X a^2 b^2 c^2 s\,(-1+s) \\ \left((\cos(v_0))^2 \sin(u_0)\, Y c^2 - \sin(v_0)\, Z b^2 \cos(v_0) - \sin(u_0)\, Y c^2\right) \cos(u_0) \\ +4\, a^2 \left(\begin{array}{c} \left(\begin{array}{c} \left(\left((s-1/2)^2\, b^2 - 1/4\, Y^2\right) a^2 - (s-1/2)^2\, b^2 X^2\right) c^4 \\ +\,\left((Y^2 - Z^2 - b^2)\, a^2 + X^2 b^2\right)(s-1/2)^2\, b^2 c^2 + 1/4\, Z^2 a^2 b^4 \end{array} \right) \\ (\cos(v_0))^2 - 2\,\sin(u_0)\sin(v_0)\, Y Z a^2 b^2 c^2 s\,(-1+s)\cos(v_0) \\ +\left(\begin{array}{c} \left(\left(-\,(s-1/2)^2\, b^2 + 1/4\, Y^2\right) a^2 + (s-1/2)^2\, b^2 X^2\right) \\ c^2 + a^2\,(s-1/2)^2\, b^2 Z^2 \end{array} \right) c^2 \end{array} \right) \end{array} \right). \tag{24}$$

We follow the idea from Proposition 7, it indeed shows the locus surface is an ellipsoid when $s \neq \frac{1}{2}$. It is clear from (23) that the major, minor and mean axes for this locus surface depends on the parameter s.

Remarks:

1. As $s \to \frac{1}{2}$, the locus ellipsoid is getting closer to the elliptical disk bounded by the intersecting elliptical curve.
2. Similar observation can be said about the transformation L_E^h; we leave this to readers to explore.

4 Geometric Interpretation When s is Large

We recall that for a fixed s, we note that the linear transformations L_E^e or L_E^h involves the parametric equations, providing us information regarding the eigenvalues for L_E^e or L_E^h of $\{2s-1, 1, 1\}$ and their corresponding eigenvectors. In this section, we will discuss the geometric interpretation when the parameter s is is a larger value with $s > 1$.

We first make use of (20) or (11) to get the implicit equation of the locus surfaces from the implicit equation of the original surfaces. We then attempt to use a CAS to compute the eigenvectors for Q_Δ^e or Q_Δ^h respectively. However, it is too much for a CAS to compute the eigenvalues and eigenvectors for Q_Δ^e or Q_Δ^h due to the large number of parameters. Instead, we use the following two numerical Examples below, to show how we can write the locus surface for an ellipsoid or a hyperboloids with two sheets in standard forms as an application of the Principal axis theorem [6]. These examples confirm our conjecture that the locus for an ellipsoid when s is large, will be another long ellipsoid containing the original ellipsoid.

Example 14 *Consider the our locus surface for the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

with $s = 20, a = 5, b = 4, c = 3$ *and the fixed point is at the infinity when* $u_0 = \frac{\pi}{3}, v_0 = \frac{\pi}{4}$*. The numerical approximation of* Q^e_Δ *is*

$$\begin{bmatrix} 135.4334986093818 & -23.18377445571889 & -47.5916743923231 & 0 \\ -23.18377445571889 & 162.2570698929334 & -128.7987469762161 & 0 \\ -47.5916743923231 & -128.7987469762161 & 135.6018089315383 & 0 \\ 0 & 0 & 0 & -3600 \end{bmatrix}.$$

The equation $XQ^e_\Delta X^T = 0$ *of the locus ellipsoid becomes*

$$135.4334986x^2 - 46.36754892xy - 95.18334882xz + 162.2570699y^2 - 257.597494yz + 135.6018089z^2 - 3600 = 0.$$

Let us note that computing eigenvalues and eigenvectors of a matrix is subject to numerical errors. In fact, some built in functions implemented to this end may produce weird results (for example, complex solutions for a real symmetric matrix). So, using a suitable built in function to approximate the eigenvalues and eigenvectors of matrix Q^e_Δ (in this case eigens_by_jacobi built in maxima function) we got,

$$\begin{aligned} \lambda_1 &= 0.1987812414007027, \\ \lambda_2 &= 153.0821898482854, \\ \lambda_3 &= 280.0114063441675, \\ \lambda_4 &= -3600. \end{aligned}$$

The corresponding unit eigenvectors are written as column vectors respectively below

$$\begin{bmatrix} 0.3537757151666001 & -0.9290517690585807 & -0.1081922075173726 & 0 \\ 0.6124527321391638 & 0.3175226342422298 & -0.7239344083818289 & 0 \\ 0.7069260175249132 & 0.1898477999688088 & 0.6813320912692791 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} w1 & w2 & w3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using the rotated system of coordinates $\tilde{x}$–$\tilde{y}$–$\tilde{z}$ determined by eigenvectors w_1, w_2, w_3, we got the equation of the locus ellipsoid in standard form $\lambda_1\tilde{x}^2 + \lambda_2\tilde{y}^2 + \lambda_3\tilde{z}^2 = -\lambda_4$, or

$$\frac{\tilde{x}^2}{\left(\sqrt{\frac{-\lambda_4}{\lambda_1}}\right)^2} + \frac{\tilde{y}^2}{\left(\sqrt{\frac{-\lambda_4}{\lambda_2}}\right)^2} + \frac{\tilde{z}^2}{\left(\sqrt{\frac{-\lambda_4}{\lambda_3}}\right)^2} = 1. \tag{25}$$

Note that

$$\sqrt{\frac{-\lambda_4}{\lambda_1}} = 134.5747405338178 > \sqrt{\frac{-\lambda_4}{\lambda_2}} = 4.849410151798864 > \sqrt{\frac{-\lambda_4}{\lambda_3}} = 3.585612795294109.$$

It follows from the direct calculations using Maxima [4] in [S2] that the angle between w_1 and eigenvector $v_1 = [\cos u_0 \tan v_0, \sin u_0 \tan v_0, 1]^t$ for L^e_E. is equal to 0.0002975755987288942 radians. Therefore, we obtain the longest major semi–axis length in this case to be 134.5747559614472 as shown in the following Figure 3. We note that the CAS [3] does confirm the numeric computations when number of Digits is increased from default 10 to 15, see [S2.1].

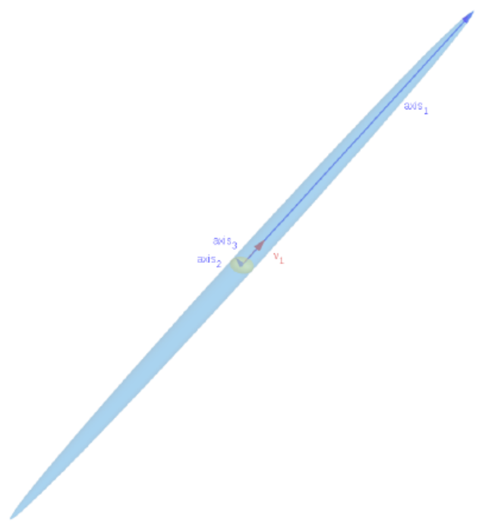

Figure 3. Locus ellipsoid when $s = 20, a = 5, b = 4, c = 3, u_0 = \frac{\pi}{3}$ and $v_0 = \frac{\pi}{4}$

This example indeed shows that the locus ellipsoid surface gets longer as s increases.

Remarks:

1. Example 13 suggests that when s gets large, the eigenvector corresponds to the longest major semi–axis for the locus surface, after being written in the standard form (25), will approach the eigenvector $v_1 = [\cos u_0 \tan v_0, \sin u_0 \tan v_0, 1]^t$ from L_E^e, as expected.
2. In the preceding Example 13, the angle between the eigenvector of the locus ellipsoid w_1 and eigenvector $v_1 = [\cos u_0 \tan v_0, \sin u_0 \tan v_0, 1]^t$ for L_E^e. is approximately 0.0002975755987288942 radians when $s = 20$. It is natural for readers to explore that if the tolerance of the angle between w_1 and v_1 is given, we can find the desired parameter s to satisfy the requirement. To explore further geometrically, we refer to [S3].

4.1 Locus surface for a hyperboloid with two sheets when s is large

Example 15 *Consider the our locus surface for the hyperboloid of two sheets*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1,$$

with $s = 20, a = 5, b = 4, c = 3$ *and the fixed point is at the infinity when* $u_0 = \frac{\pi}{6}, v_0 = \frac{\pi}{3}$ *. The numerical approximation of* Q^h_Δ *is*

$$\begin{bmatrix} -358.6989266108146 & -453.4896259352358 & 930.9239381629927 & 0 \\ -453.4896259352358 & -184.0974337674445 & 839.795603583711 & 0 \\ 930.9239381629927 & 839.795603583711 & -2123.933218812494 & 0 \\ 0 & 0 & 0 & 3600 \end{bmatrix}.$$

It follows from [S5] and [S5.1] that the equation $X\ Q^h_\Delta X^T = 0$ *of the locus hyperboloid with two sheets becomes*

$$12650661x^2+31987514.3142xy-65664000xz+6492782.8125y^2-59236137.6189yz+74907275z^2-12696547.5=0.$$

The approximate eigenvalues and eigenvectors of matrix Q^h_Δ are,

$$\begin{aligned} \lambda_1 &= 0.01525397670016254, \\ \lambda_2 &= 195.179651674006, \\ \lambda_3 &= -2861.924484841458, \\ \lambda_4 &= 3600. \end{aligned}$$

The corresponding unit eigenvectors are written as column vectors respectively below

$$\begin{bmatrix} 0.7500323743583207 & -0.5405430593677909 & -0.3811359841103027 & 0 \\ 0.4330148784900295 & 0.8369014258696608 & -0.3348045973154995 & 0 \\ 0.4999495498754701 & 0.08607673522292496 & 0.8617663507196585 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} w1 & w2 & w3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using the rotated system of coordinates $\tilde{x}$–$\tilde{y}$–$\tilde{z}$ determined by eigenvectors w_1, w_2, w_3, we got the equation of the locus hyperboloid with two sheets in standard form $\lambda_1\tilde{x}^2+\lambda_2\tilde{y}^2+\lambda_3\tilde{z}^2 = -\lambda_4$, or

$$\frac{\tilde{x}^2}{\left(\sqrt{\frac{\lambda_4}{\lambda_1}}\right)^2} + \frac{\tilde{y}^2}{\left(\sqrt{\frac{\lambda_4}{\lambda_2}}\right)^2} - \frac{\tilde{z}^2}{\left(\sqrt{\frac{\lambda_4}{-\lambda_3}}\right)^2} = -1. \tag{26}$$

Note that

$$\sqrt{\frac{\lambda_4}{\lambda_1}} = 485.8024615565034 > \sqrt{\frac{\lambda_4}{\lambda_2}} = 4.294711361002595 > \sqrt{\frac{\lambda_4}{-\lambda_3}} = 1.121559104702658.$$

It follows from direct calculations using [4] in exploration [S5] that the angle between w_1 and eigenvector $v_1 = [\cos u_0 \tan v_0, \sin u_0 \tan v_0, 1]^t$ for L^h_E is equal to 0.0005998376145854856 radians. So, we obtain the longest major semi–axis length in this case to be 485.8024615565034 as shown in the following Figure 4. To explore further geometrically, see [S6].

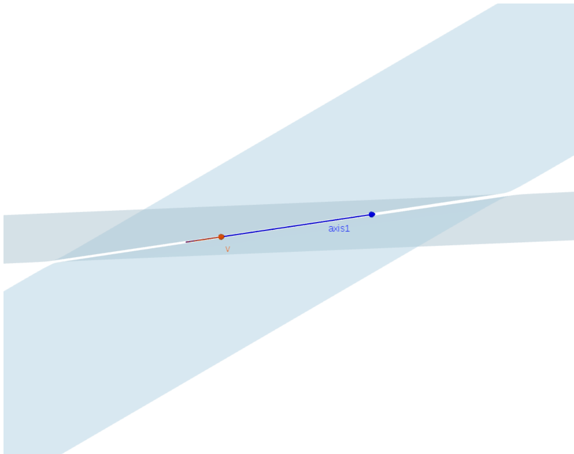

Figure 4. Locus surface for a hyperboloid with two sheets when $s = 20$

Remarks:

1. After the locus surface for an ellipsoid is written in standard quadratic form, when s increases, the length of the major axis, $\sqrt{\frac{-\lambda_4}{\lambda_1}}$, also increases.

2. As we see from Example 13 for the ellipsoid when $s = 20$, the eigenvector corresponding to the largest major axis will be close to $v_1 = \begin{bmatrix} \cos u_0 \tan v_0 \\ \sin u_0 \tan v_0 \\ 1 \\ 0 \end{bmatrix}$. This observation is consistent with the eigenvalue, $2s-1$, and eigenvector, $v_1 = [\cos u_0 \tan v_0, \sin u_0 \tan v_0, 1]^t$, for L_E^e. In other words, when s increases, the major axis for the ellipsoid converges to the direction of the eigenvector v_1.

3. Similar observations can be done for the Locus surface of a hyperboloids with two sheets when s is large, say $s = 20$. We leave it to the readers to explore this situation in more detail. In the following, we explore what happens to the locus surfaces when $s \to \infty$.

4.2 Locus surfaces in the limit as the parameter $s \to \infty$

It is expected that it is too complicated for a CAS to compute the eigenvectors for Q_Δ^e and Q_Δ^h, respectively, directly if we let $s \to \infty$. Instead, we take $s \to \infty$ for each entry of

$$\begin{bmatrix} A & B/2 & C/2 & 0 \\ B/2 & D & E/2 & 0 \\ C/2 & E/2 & F & 0 \\ 0 & 0 & 0 & J \end{bmatrix} \tag{27}$$

for Q^e_Δ and Q^h_Δ respectively first, and find the corresponding eigenvalues and eigenvectors. We label such 4×4 matrices as $(Q^e_\Delta)'$ and $\left(Q^h_\Delta\right)'$ respectively. Accordingly, we label the 3×3 matrix

$$\begin{bmatrix} A & B/2 & C/2 \\ B/2 & D & E/2 \\ C/2 & E/2 & F \end{bmatrix} \tag{28}$$

of $(Q^e_\Delta)'$ and $\left(Q^h_\Delta\right)'$ by $(Q^e_\Delta)''$ and $\left(Q^h_\Delta\right)''$ respectively. Since we are letting the parameter $s \to \infty$, in order to categorize the locus surfaces correctly, we need to examine the conditions mentioned in [8] carefully. We see the ranks of $(Q^e_\Delta)'$ and $\left(Q^h_\Delta\right)'$are now 3 instead of the original 4. (For calculations of the ranks, please see [S2.1] and [S5.1] respectively.)

1. For the ellipsoid case of $(Q^e_\Delta)'$, we refer to the computations obtained in [S2] and [S2.1] that the product of two non-zeros eigenvalues to be

$$\frac{a^4b^4c^4}{\left(a^2c^2\sin^2(u_0)+\cos^2(u_0)b^2c^2\right)\sin^2(v_0)+\cos^2(v_0)a^2b^2} > 0, \tag{29}$$

   (a) It follows from [8] that the locus surface in this case is called an elliptical cylinder. We can see that the eigenvector $X = \begin{bmatrix} \cos u_0 \tan v_0 \\ \sin u_0 \tan v_0 \\ 1 \\ 0 \end{bmatrix}$ corresponds to the eigenvalue 0. Since we have $(Q^e_\Delta)' X = 0$, this transformation sends the vector X to the origin, and consequently the locus surface becomes an open ended elliptical cylinder. We note that the base is an ellipse which is spanned by the other two eigenvectors.

   (b) We depict the locus elliptical cylinder (yellow) when $a = 5, b = 4, c = 3$, with $u_0 = \frac{\pi}{6}, v_0 = \frac{\pi}{3}$ together with the original ellipsoid (blue) in Figures 5(a) and 5(b). The observation of the rank of $(Q^e_\Delta)'$ implies the locus surface to be an elliptical cylinder, which is consistent with the observation from [8]. For exploration, see

[S2.1] and [S3].

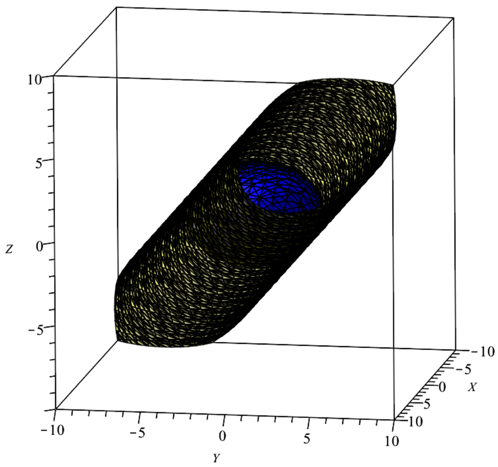

Figure 5(a). The original ellipsoid and its locus when $s \to \infty$.

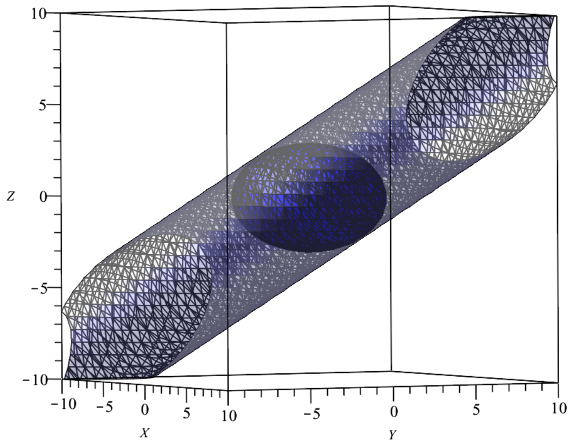

Figure 5(b). The ellipsoid and its locus are tangent to each other

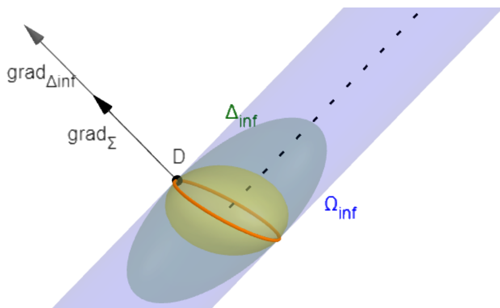

Figure 5(c). Ellipsoid and when locus surfaces $s \to \infty$

(c) In [S2], we show that the cross product of the respective gradients, of the Σ and its locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ at the point on the intersecting curve, is 0. In [S3.1], we simulate the locus ellipsoid surface in green will go toward the ellipsoid cylinder when $s \to \infty$. Moreover, readers can visualize that the respective gradients of the Σ and its locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$, at a point on the intersecting curve, are parallel.

(d) Since the locus surface becomes a different geometric structure when $s \to \infty$, we leave it to the readers to verify that the property of Proposition 3 does hold as we see in Figure 5(b). In the exploration [S3.1], readers can adjust the slider for s to simulate that the locus ellipsoid surface in green will go toward the ellipsoid cylinder (in blue) when $s \to \infty$ as shown in Figure 5(c).

(e) Furthermore, as we see from (23) that the locus surface is not defined when $s = \frac{1}{2}$. In fact, the locus becomes the elliptical disk bounded by intersecting elliptical curve as we have shown in (10) that the elliptical disk is spanned by $\{v_2, v_3\}$. We also note that the eigenvalue is 0 when $s = \frac{1}{2}$ for the eigenvector v_1. In other words, we are looking at

$$L_E^e v_1 = 0. \tag{30}$$

In this case, we are sending v_1 back to the origin. Therefore, we obtain a two dimensional elliptical disk in this case.

2. As for the case of $\left(Q_\Delta^h\right)'$, we follow the ideas from [8], and we are able to compute the ranks for $\left(Q_\Delta^h\right)''$and $\left(Q_\Delta^h\right)'$ to be 2 and 3 respectively, and det $\left(\left(Q_\Delta^h\right)'\right) = 0$ using [3], see [S5.1]. However, the signs of the two non-zeros eigenvalues for $\left(Q_\Delta^h\right)''$, λ_1 and λ_2 (see [S5.1]), cannot be determined to be the same or not.

   (a) For example, if we use $a = 5, b = 4, c = 3$, $u_0 = \frac{\pi}{6}$ and $v_0 = \frac{\pi}{3}$, we see $\lambda_1 \cdot \lambda_2 = -558921.8353 < 0$. On the other hand, if we use $a = 5, b = 4, c = 3$, $u_0 = \frac{\pi}{6}$ and $v_0 = \frac{\pi}{3}$, we see $\lambda_1 \cdot \lambda_2 = 109946.9778 > 0$. (See [S5.1] for computations.) Therefore, it is inconclusive to categorize the locus surface for a hyperboloid with two sheets, when $s \to \infty$ and the fixed point is at an infinity.

   (b) If we use $a = 5, b = 4, c = 3$, $u_0 = \frac{\pi}{6}$ and $v_0 = \frac{\pi}{3}$, when $\lambda_1 \cdot \lambda_2 < 0$, as demonstration. The locus surface is categorized as a hyperbolic cylinder. We see that the eigenvector $X = \begin{bmatrix} \cos u_0 \tan v_0 \\ \sin u_0 \tan v_0 \\ 1 \\ 0 \end{bmatrix}$ corresponds to the eigenvalue 0. Hence, $\left(Q_\Delta^h\right)' X = 0$ means that the transformation sends X to the origin. In this case, we depict the locus of hyperbolic cylinder (since $\lambda_1 \cdot \lambda_2 < 0$) in yellow, and the original hyperboloid with two sheets in red in Figure 6(a).

   (c) We show the original surface in red and its corresponding locus in yellow in Figure 6(a). Furthermore, we show the intersecting curves, in blue, between these two surfaces in Figure 6(b). We refer readers to [S5.1] for further exploration. We also leave it to the readers to verify that the property of Proposition 5 does hold when $s \to \infty$ and the surfaces Σ and its locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ intersect at a hyperbola. We refer readers to [S5.1], [S6] and [S6.1] for further explorations. In [S6.1], readers can visualize that the respective gradients of the Σ and its locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$, at points on the intersecting hyperbolic curve, are parallel. Equivalently, we show in [S5] that the cross product of the respective gradients, of the Σ and its locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ at the points on the intersecting hyperbolic curve, is 0.

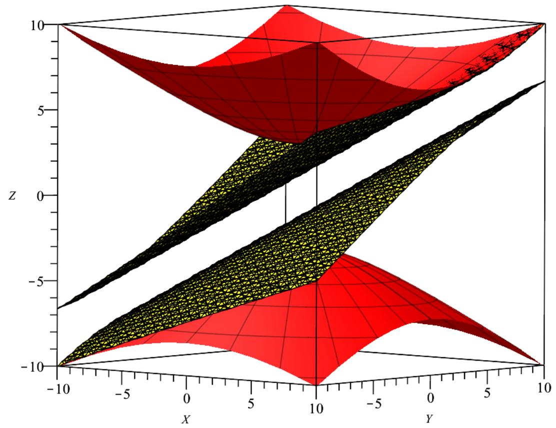

Figure 6(a). Locus hyperbolic cylinder when $u_0 = \frac{\pi}{6}$, $v_0 = \frac{\pi}{3}$ and $s \to \infty$.

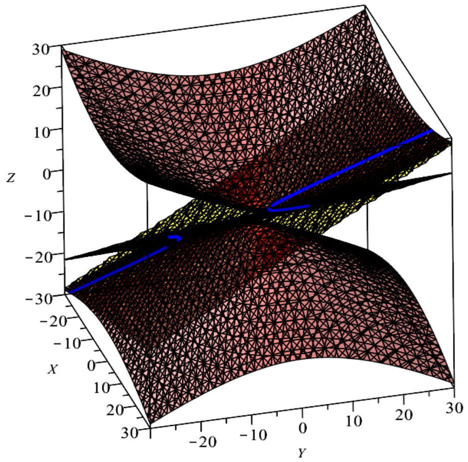

Figure 6(b). Locus hyperbolic cylinder when $u_0 = \frac{\pi}{6}$, $v_0 = \frac{\pi}{3}$ and $s \to \infty$.

5 Conclusion

The antipodal locus problems when the fixed point is at an infinity discussed in this paper definitely provides interesting areas in projective geometry, see [2], and it can lead to further explorations in algebraic geometry. As we have discussed in this paper that the ellipsoid Σ and its locus ellipsoid $\Delta_{\text{inf}}(s, u_0, v_0)$ intersect at an elliptic curve when the direction (u_0, v_0) for the fixed A at infinity and s are given. One may ask an inverse question as follows: Suppose the original ellipsoid Σ is given, the linear transformation L_E^e of 9 is applied when A is chosen. Then the locus ellipsoid $\Delta_{\text{inf}}(s, u_0, v_0)$ is found. If we rotate the intersecting plane P, which contains the elliptic intersecting curve, by keeping the mean axis fixed, and tilting the minor axis towards the major axis for $\Delta_{\text{inf}}(s, u_0, v_0)$. How can we choose the fixed point A so that the new plane P' intersects the ellipsoid in a round circle.

We also discussed the shape of a locus when the parameter s is large. Intuitively, the locus surface for an ellipsoid will be a larger ellipsoid when s gets larger. Consequently, we see that when $s \to \infty$, the locus surfaces for an ellipsoid or a hyperboloid with two sheets will change their topological structures as we saw in Section 4.2. Our investigations of these situations with various technological tools were critical to the development of our intuition and conjectures that were the foundation of our subsequent more rigorous analytical conclusions. Here we have gained geometric intuitions while using a DGS such as [1]. In the meantime, we use a CAS such as [3] or [4] for verifying that our analytical solutions are consistent with our initial intuitions. Many of our solutions are accessible to students from high school. Others require more advanced mathematics such as university levels, and are excellent examples for professional trainings for future teachers.

It is a delight to see how a simple college entrance exam from China, after being explored with technological tools (see [9]), has evolved into interesting problems in different fields, including projective geometry, differential geometry (see [11]), and possibly algebraic geometry. Evolving technological tools definitely have made mathematics fun and accessible on one hand, but they also allow the exploration of more challenging and theoretical mathematics. We hope that when mathematics is made more accessible to students, it is possible more students will be inspired to investigate problems ranging from the simple to the more challenging. We do not expect that exam-oriented curricula will change in the short term. However, encouraging a greater interest in mathematics for students, and in particular providing them with the technological tools to solve challenging and intricate problems beyond the reach of pencil-and-paper, is an important step for cultivating creativity and innovation.

6 Acknowledgements

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7 Supplementary Electronic Materials

[S1] wxMaxima worksheet for Methods 1 and 2 in Section 2.

[S2] wxMaxima worksheet for ellipsoid case in Sections 3 and 4.

[S2.1] Maple worksheet for ellipsoid case in Sections 3 and 4.

[S3] GeoGebra worksheet for ellipsoid case in Sections 3 and 4.

[S3.1] GeoGebra worksheet for ellipsoid case in Section 4.2.

[S4] Maple worksheet for hyperboloid case in Section 3

[S5] wxMaxima worksheet for hyperboloid case in Sections 3 and 4.

[S5.1] Maple worksheet for hyperboloid case in Sections 3 and 4.

[S6] GeoGebra worksheet for hyperboloid case in Sections 3 and 4.

[S6.1] GeoGebra worksheet for hyperboloid case in Section 4.2.

[S7] wxMaxima worksheet for hyperboloid case in Example 7.

[S7.1] GeoGebra worksheet for Example 7.

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